

A statistical approach to the theory of the mean field

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1. – Introduction

The study of nuclear reactions has prompted the delineation of fundamental principles whose importance extends beyond nuclear physics, from where they were originally inferred, but reaches vast sectors of physics in general and perhaps of other fields as well.

Nuclear reactions are of central interest because they deal with strong interactions, for which perturbative approaches are of little value. Indeed, these strong forces, acting at short distances between the nucleons, entail large matrix elements to excited states of the nucleus lying at high energy: this occurrence in turn leads to a poor convergence of the diagrammatic expansion. In reaction theory the above problem finds a solution in the recognition that once an average value (to be appropriately defined) is subtracted from these matrix elements, then the resulting quantities are *random*. Actually, we believe these results to hold not only for the continuum, but for the nuclear bound states as well and also for other systems beside the nuclear one. In other words we conjecture that many strongly interacting systems in states of high excitations are *chaotic*.

In this conceptual framework an approach is here presented aiming at the derivation of the nuclear mean field (MF), binding the nucleons together in the nuclear ground

state, from statistical considerations. In general, the mean field happens to be a good description for a variety of many-body systems, whose constituents can interact either with strong or weak forces, of long or short range. It is thus natural to search for the features, common to the various systems, ensuring the validity of the mean field approximation. In line with the above speculations we hold the view that the statistical aspect is the common feature linking all these different systems together.

Accordingly, we base our treatment of the MF on *two* propositions:

- a) the MF can be obtained through an energy average that smooths out short time events as, e. g., those corresponding to the violent short-range nucleon-nucleon encounters in nuclei;
- b) the matrix elements of the residual interactions, namely the force left out after the average interaction has been removed, are random with zero average value.

As we shall see these two elements suffice to obtain explicit expressions for both the MF and the fluctuations (the *error*) away from the average.

Clearly the error should be small for the MF to be meaningful. In this connection it turns out that, because of item b), the average error actually vanishes. However, the average of the *square* of the error is not vanishing and its magnitude we wish to assess. For this purpose, we expand the fluctuations of the MF in terms of excitations of increasing complexities, each class of excitations providing a contribution to the error. Notably, a general condition for the convergence of this expansion (which is a finite one in the case of atomic nuclei) can be derived; in fact, a parameter can be identified determining the relative importance of each contribution to the error.

It is finally worth emphasizing that the theory here developed is, in the one hand, quite general in the sense that no assumptions on the nature of the system being studied are made, and, on the other, closely inspired from, in fact almost identical with, the statistical theory of nuclear reactions. Indeed it provides, as the latter does, an alternative method to account for the contributions to physical observables stemming from processes slowly and rapidly varying with the energy. It differs from the customary approaches which are based on the explicit consideration of the Pauli and dynamical correlations, the latter typically providing rapidly varying contributions.

In order to get an appreciation on how the method is practically implemented an exploratory calculation of the ground state energy of an infinite, interacting Fermi gas, i. e. nuclear matter, will be discussed at the end of this lecture. For this system a reasonable estimate of the error is obtained. Furthermore, and significantly, the residual interaction appears to be drastically reduced with respect to the bare one, an occurrence directly related to the statistical nature of the present approach.

The present lecture, which is based on Ref. [1], is organized as follows. In sect. **2** the formalism is revisited, — in particular the partition of the Hilbert space into the *P* and *Q* sectors, — and the related equations are discussed, while in sect. **3** a precise definition of the *energy average* is provided. The latter is performed with a distribution, characterized by a single parameter ϵ , amounting to shift the position of the poles corresponding to

the excited states of the system away from the ground state energy. In the same section we pave the way for setting up a suitable formalism to account for the corrections (the error) to the mean field energy.

In sect. 4 the expression for the corrections to the mean field energy \bar{E}_0 is given in the framework of the expansion, above referred to, built out of successive 2p–2h excitations.

In sect. 5 we deal with the issue of introducing an operator, projecting into a smooth P -space, simple enough to allow the present approach to be worked out for nuclear matter. We thus restrict the P -space, — also with the aim of rendering transparent the comparison between the mean field energy as given by the Hartree-Fock (HF) approach and by the present theory, — just to a simple state: The HF determinant. With this choice for P , we are able to establish an expression connecting the true ground state E , the mean field \bar{E}_0 and the Hartree-Fock E_{HF} energies. Importantly, \bar{E}_0 is shown to be *always* lower than E_{HF} and it turns out to be proportional to the variance, to be later defined, of the residual effective interaction V among nucleons, which, in our context, is precisely defined.

In sect. 6 we address the issue of the occupation probability S^2 (spectroscopic factor) of the ground state of nuclear matter as defined by the structure of our P -space. We are able to deduce, within our scheme, an expression for S^2 in terms of E , \bar{E}_0 and E_{HF} , and moreover embodying the energy derivative of V .

Finally, in sect. 7 a scheme which permits numerical predictions of the present approach is developed. It refers to the infinite Fermi gas, namely nuclear matter.

For sake both of simplicity and of illustration, we account for the first corrections only and, even so, a free parameter α should be introduced to obtain an estimate of their size. Two systems of two equations each can then be set up by coupling the equation for the mean field energy to the one for the fluctuations. By solving the systems and with a convenient choice of the two parameters entering into the present approach, namely ϵ and α , results consistent with the empirical features of nuclear matter, as derived through an extrapolation from finite nuclei, are obtained. At the same time an estimate for the fluctuations of the mean field energy is provided. These goals are achieved with a residual effective interaction which, as previously emphasized, is lowered by about three orders of magnitude with respect to the bare one and yields a spectroscopic factor not in conflict with other independent estimates [3].

In the concluding section the approach is summarized and possible improvements discussed.

2. – Formalism

In this section we shortly present the formalism originally introduced in Ref. [4]. Let H be the nuclear hamiltonian entering into the Schrödinger equation

$$(2.1) \quad H\Psi = E\Psi.$$

Let P be the hermitian operator projecting the nuclear ground state into the Hilbert subspace of functions associated with the low momentum transfer physics and Q the

operator complementing P . Clearly

$$(2.2) \quad P^2 = P, \quad Q^2 = Q, \quad PQ = 0, \quad P + Q = \mathbf{1}.$$

It is then proved that the pair of equations

$$(2.3) \quad (E - H_{PP})(P\Psi) = H_{PQ}(Q\Psi),$$

and

$$(2.4) \quad (E - H_{QQ})(Q\Psi) = H_{QP}(P\Psi),$$

are equivalent to (2.1). In the above formulas the following shorthand notations have been introduced, viz.

$$(2.5) \quad H_{PP} = PHP, \quad H_{QQ} = QHQ, \quad H_{PQ} = PHQ, \quad H_{QP} = QHP.$$

Now a direct way to proceed in order to obtain $P\Psi$ is to get from (2.3)

$$(2.6) \quad (P\Psi) = \frac{1}{E - H_{PP}} H_{PQ}(Q\Psi)$$

and from (2.4)

$$(2.7) \quad (Q\Psi) = \frac{1}{E - H_{QQ}} H_{QP}(P\Psi).$$

Inserting then (2.7) in (2.6) one obtains

$$(2.8) \quad (P\Psi) = \frac{1}{E - H_{PP}} H_{PQ} \frac{1}{E - H_{QQ}} H_{QP}(Q\Psi)$$

or, equivalently, after multiplication by $E - H_{PP}$,

$$(2.9) \quad \left(E - H_{PP} - H_{PQ} \frac{1}{E - H_{QQ}} H_{QP} \right) (P\Psi) = 0,$$

namely the equation obeyed by the component of Ψ in the P -space. A more general procedure is however required should a solution of the equation

$$(2.10) \quad (E - H_{PP})\Psi_0 = 0$$

exists.

In such a case, instead of eqs. (2.6) and (2.7), one has

$$(2.11) \quad (P\Psi) = \Psi_0 + \frac{1}{E - H_{PP}} H_{PQ}(Q\Psi)$$

and

$$(2.12) \quad (Q\Psi) = \frac{1}{E - H_{QQ} - W_{QQ}} H_{QP}\Psi_0 \equiv \frac{1}{e_Q} H_{QP}\Psi_0,$$

where

$$(2.13) \quad W_{QQ} = H_{QP} \frac{1}{E - H_{PP}} H_{PQ}$$

and

$$(2.14) \quad e_Q = E - H_{QQ} - W_{QQ}.$$

The above equations allow then one to recast (2.3) as follows

$$(2.15) \quad (E - H_{PP})(P\Psi) = H_{PQ} \frac{1}{e_Q} H_{QP} \Psi_0$$

and to express Ψ_0 according to

$$(2.16) \quad \Psi_0 = \frac{1}{1 + \frac{1}{E - H_{PP}} H_{PQ} \frac{1}{e_Q} H_{QP}} (P\Psi).$$

The combination of the two above equations leads in turn to

$$(2.17) \quad (E - H_{PP})(P\Psi) = H_{PQ} \frac{1}{e_Q} H_{QP} \frac{1}{1 + \frac{1}{E - H_{PP}} H_{PQ} \frac{1}{e_Q} H_{QP}} (P\Psi),$$

where no trace is left of Ψ_0 . Now using the operator identity

$$(2.18) \quad B \frac{1}{1 + CAB} = \frac{1}{1 + BCA} B,$$

with the identifications

$$(2.19) \quad A = \frac{1}{e_Q}, \quad B = H_{QP}, \quad C = \frac{1}{E - H_{PP}} H_{PQ},$$

one obtains

$$(2.20) \quad \begin{aligned} (E - H_{PP})(P\Psi) &= H_{PQ} \frac{1}{e_Q} \frac{1}{1 + H_{QP} \frac{1}{E - H_{PP}} H_{PQ} \frac{1}{e_Q}} H_{QP} (P\Psi) \\ &= H_{PQ} \frac{1}{e_Q} \frac{1}{1 + W_{QQ} \frac{1}{e_Q}} H_{QP} (P\Psi). \end{aligned}$$

The obvious identity

$$(2.21) \quad \frac{1}{e_Q} \frac{1}{1 + W_{QQ} \frac{1}{e_Q}} = \frac{1}{\left(\frac{1}{e_Q} \right)^{-1} + W_{QQ}}$$

leads finally to

$$(2.22) \quad \left(E - H_{PP} - H_{PQ} \frac{1}{\left(\frac{1}{e_Q} \right)^{-1} + W_{QQ}} H_{QP} \right) (P\Psi) = 0,$$

which is entirely equivalent to (2.9).

It is important to note that the equation obeyed by $P\Psi$, being associated with the intricate many-body operator

$$(2.23) \quad \mathcal{H} = H_{PP} + H_{PQ} \frac{1}{\left(\frac{1}{e_Q} \right)^{-1} + W_{QQ}} H_{QP},$$

is not an eigenvalue equation in the usual sense. Indeed, the dependence upon E , the exact ground state energy of the system, is non-linear, since the latter appears also in the propagator in the Q -space in (2.22).

3. – Averaging upon the energy

The partition of the Hilbert space into a P and a Q sector should be performed in conformity to the principle of including into the P -space the wave functions with the simplest structure, namely those with a smooth spatial dependence, and in the Q -space the wave functions of greater intricacy. The actual Ψ of the system should of course be viewed as a linear superposition of components with all the possible degrees of complexity.

The problem then is: how to account for the average impact on the P -space of the Q -space wave functions, ignoring the detailed behaviour of the latter? To deal with this question an averaging procedure should first be prescribed. For this purpose we treat the energy E as a variable upon which both the wave function and the matrix elements depend. Clearly, the components of the wave function varying most rapidly with E lie in the Q -space: thus performing an average over the energy amounts to smoothing out the behaviour of $Q\Psi$ which can then be taken into account in the determination of the mean field. The latter rules the gentle physics taking place in the P -space.

However, by replacing $(Q\Psi)$ with $\langle Q\Psi \rangle$ we also change at the same time $(P\Psi)$ into $\langle P\Psi \rangle$ and E into, say, \bar{E}_0 (the angular brackets meaning, of course, energy averaging).

The system equivalent to the Schroedinger equation will now accordingly read

$$(3.1) \quad \langle P\Psi \rangle = \tilde{\Psi}_0 + \frac{1}{\bar{E}_0 - H_{PP}} H_{PQ} \langle Q\Psi \rangle,$$

where

$$(3.2) \quad (\bar{E}_0 - H_{PP}) \tilde{\Psi}_0 = 0$$

and

$$(3.3) \quad \langle Q\Psi \rangle = \langle \frac{1}{e_Q} \rangle H_{QP} \tilde{\Psi}_0.$$

Inserting (3.3) into (3.1) yields

$$(3.4) \quad (\bar{E}_0 - H_{PP}) \langle P\Psi \rangle = H_{PQ} \langle \frac{1}{e_Q} \rangle H_{QP} \tilde{\Psi}_0.$$

Following then the same steps as before the equation

$$(3.5) \quad \left(\bar{E}_0 - H_{PP} - H_{PQ} \frac{1}{\langle \frac{1}{e_Q} \rangle + W_{QQ}} H_{QP} \right) \langle P\Psi \rangle \equiv (\bar{E}_0 - \bar{\mathcal{H}}) \langle P\Psi \rangle = 0$$

is obtained: it clearly corresponds to equation (2.22) when a suitable energy average has been performed.

We should now face the problem of specifying the energy average. For this purpose a smoothing function $\rho(E, \bar{E}_0)$ with the property

$$(3.6) \quad \int \rho(E, \bar{E}_0) dE = 1$$

should be introduced. Then averaging a function $f(E)$ means to perform the integral

$$(3.7) \quad \langle f \rangle = \int \rho(E, \bar{E}_0) f(E) dE$$

which must be a real quantity since we are dealing with bound states. A smoothing function obeying the above conditions is

$$(3.8) \quad \rho(E, \bar{E}_0) = \frac{1}{2\pi i} \frac{1}{E - (\bar{E}_0 - \epsilon)}.$$

In this case the integration in eq. (3.7) is performed along a path coinciding with the real axis ReE with a small semi-circle described positively about the singularity $(\bar{E}_0 - \epsilon)$. If $f(E)$ is bounded at infinity sufficiently strongly, the Cauchy's integral formula can be applied so that the condition of eq. (3.6) is satisfied and eq. (3.7) becomes

$$(3.9) \quad \langle f \rangle = f(\bar{E}_0 - \epsilon).$$

In the present approach ϵ is an empirical parameter essentially measuring the range over which the average is taken. In other words, for a sufficiently large value the energy of the states within ϵ can be neglected and it is in this sense that an average is performed.

At this point the system corresponding to the original Schroedinger equation can be cast into the form (see also Kawai, Kerman and McVoy [5])

$$(3.10) \quad (E - \bar{\mathcal{H}})(P\Psi) = V_{PQ}(Q\Psi)$$

$$(3.11) \quad (E - H_{QQ})(Q\Psi) = V_{QP}(P\Psi)$$

where

$$(3.12) \quad \bar{\mathcal{H}} = H_{PP} + V_{PQ} V_{QP} \frac{1}{\bar{E}_0 - \epsilon - E}$$

and

$$(3.13) \quad V_{PQ} = H_{PQ} \sqrt{\frac{\bar{E}_0 - \epsilon - E}{\bar{E}_0 - \epsilon - H_{QQ}}},$$

$$(3.14) \quad V_{QP} = \sqrt{\frac{\bar{E}_0 - \epsilon - E}{\bar{E}_0 - \epsilon - H_{QQ}}} H_{QP}.$$

Note that eqs. (3.10) and (3.11), while having the same structure as the original eqs. (2.3) and (2.4), display a potential, coupling $(P\Psi)$ and $(Q\Psi)$, which is V_{PQ} rather than H_{PQ} . The strength of the coupling is thus considerably reduced by roughly $(\epsilon/H_{QQ})^{\frac{1}{2}}$ which is much less than one. This in turn entails that H_{PP} and $\bar{\mathcal{H}}$ are of the same order of magnitude.

Now the spectral decomposition of the operator $1/(E - \bar{\mathcal{H}})$ can be performed in terms of eigenfunctions of the hermitian operator $\bar{\mathcal{H}}$, namely

$$(3.15) \quad \bar{\mathcal{H}}\Phi_n = \bar{E}_n \Phi_n.$$

It reads

$$(3.16) \quad \begin{aligned} (P\Psi) &= \frac{1}{E - \bar{\mathcal{H}}} V_{PQ} (Q\Psi) = \sum_n \frac{|\Phi_n\rangle \langle \Phi_n|}{E - \bar{E}_n} V_{PQ} (Q\Psi) \\ &= |\Phi_0\rangle \frac{\langle \Phi_0 | V_{PQ} | Q\Psi \rangle}{E - \bar{E}_0} + \left(\frac{1}{E - \bar{\mathcal{H}}} \right)' V_{PQ} (Q\Psi), \end{aligned}$$

the prime on $(1/(E - \bar{\mathcal{H}}))$ signifying that the lowest eigenfunction Φ_0 is to be excluded.

From eq. (3.16) it follows

$$(3.17) \quad \langle \Phi_0 | P\Psi \rangle = \frac{\langle \Phi_0 | V_{PQ} | Q\Psi \rangle}{E - \bar{E}_0}.$$

To obtain $(Q\Psi)$ we return to eqs. (3.10) and (3.11) and get

$$(3.18) \quad (Q\Psi) = \frac{1}{E - h_{QQ}} V_{QP} |\Phi_0\rangle \langle \Phi_0 | P\Psi \rangle,$$

where

$$(3.19) \quad h_{QQ} = H_{QQ} + \bar{W}_{QQ}$$

and

$$(3.20) \quad \bar{W}_{QQ} \equiv V_{QP} \left(\frac{1}{E - \bar{\mathcal{H}}} \right)' V_{PQ}.$$

Inserting eq. (3.18) into eq. (3.17) one finally obtains

$$(3.21) \quad E - \bar{E}_0 = \langle \Phi_0 | V_{PQ} \frac{1}{E - h_{QQ}} V_{QP} | \Phi_0 \rangle,$$

where the right hand side represents the correction to the mean field energy \bar{E}_0 .

Remarkably, the quantity $\langle \Phi_0 | P\Psi \rangle$ does not appear in eq. (3.21), the reason being related to the arbitrariness of the amplitude of Φ_0 . However, the bound state wave function Φ_0 can be normalized: our choice is $\langle \Phi_0 | P\Psi \rangle = 1$.

4. – Corrections to the energy of the mean field

We now search for an expression yielding the corrections to the ground state energy \bar{E}_0 of the shell model in the statistical framework. This is accomplished along the same line adopted in the theory of nuclear reactions [2, 6]. Accordingly (3.21) is expressed through an expansion where each contribution is not identified by a coupling constant to some power, but rather by the degree of complexity of the states of the Q -space.

One thus writes

$$(4.1) \quad \Delta E = \sum_{m=1}^r \Delta E_m,$$

where

$$(4.2) \quad \Delta E_m = \langle \Phi_0 | V_{01} G_1 V_{12} G_2 \cdots V_{m-1,m} G_m V_{m,m-1} G_{m-1} \cdots V_{10} | \Phi_0 \rangle,$$

$$(4.3) \quad V_{ij} = Q_i V Q_j,$$

and Q_j is the operator projecting into the j particles– j holes sector of the Q -space.

The above corresponds to a partition of the Q -space into r sets, each one of these embodying a specific class of excited states: thus to the k -th set are associated the k particle– k hole excitations. Actually, assuming the dominance of two-body forces in the residual interaction V (the operator connecting one set to the other), only excitations corresponding to an *even* number of particles (and holes) are actually filling the sets, as indicated in Fig. 1. The propagation of the system into the k -th set is then described by the operator G_k , which, when $k = r$, reads

$$(4.4) \quad G_r = \frac{1}{E - h_{rr}},$$

whereas for $k < r$ obeys the recurrence relation

$$(4.5) \quad G_k = \frac{1}{E - h_{kk} - V_{k,k+1} G_{k+1} V_{k+1,k}}$$

with

$$(4.6) \quad h_{kk} = H_{Q_k Q_k} + V_{Q_k P} \left(\frac{1}{E - \mathcal{H}} \right)' V_{P Q_k}.$$

Let us now, for convenience, pictorially describe a set as a box (see Fig. 1): then eq. (4.2) holds valid only if in the expansion the occupation of a box occurs when the previous box has also been occupied.

Notice also that in a nucleus the expansion (4.1) is finite since the excited states with the greatest complexity correspond to the case $r = A$, A being the mass number.

Since the mean field contains the average, the average of the corrections to the mean field vanishes. In addition because the processes occurring beyond the mean field are random, the average of the corrections to the shell model arising from any individual set vanishes as well. Therefore we should have

$$(4.7) \quad \langle \Delta E \rangle = \langle \Delta E_1 \rangle + \langle \Delta E_2 \rangle + \cdots + \langle \Delta E_A \rangle = 0$$

and

$$(4.8) \quad \langle \Delta E_n \rangle = 0.$$

However the average of the square of the corrections

$$(4.9) \quad \langle (\Delta E)^2 \rangle = \left\langle \sum_{n,m} \Delta E_n \Delta E_m \right\rangle$$

has no reason to vanish. Furthermore, owing to the randomness of the Q -space physics, the nuclear matrix elements entering into the definition of ΔE_n will also be random, hence the cancellation of the off-diagonal elements in (4.9), leaving

$$(4.10) \quad \langle (\Delta E)^2 \rangle = \left\langle \sum_n (\Delta E_n)^2 \right\rangle.$$

To ensure the fulfillment of the requirement

$$(4.11) \quad \langle \Delta E_1 \rangle = \langle \Delta E_2 \rangle = \cdots = \langle \Delta E_A \rangle = 0$$

we redefine ΔE_1 according to

$$(4.12) \quad \Delta E_1 = \langle \phi_A^0 | V_{01} G_1 V_{10} | \phi_A^0 \rangle - [\langle \phi_A^0 | V_{01} G_1 V_{10} | \phi_A^0 \rangle]_{AV},$$

and likewise for the terms with $r \neq 1$. In (4.12) ϕ_A^0 represents the nuclear mean field ground state wave function and the square brackets mean energy averaging.

We are now in a position to calculate, according to the formula (3.21), the corrections to the mean field, provided the eigenfunctions $\psi_{k,\alpha}$ and the eigenvalues $\epsilon_{k,\alpha}$ of the operator G_k^{-1} are known. We thus assume that the equation

$$(4.13) \quad G_k^{-1} \psi_{k,\alpha} = (E - \epsilon_{k,\alpha}) \psi_{k,\alpha}$$

can be solved for any value of the index k .

Let us then first consider ΔE_1 :

$$(4.14) \quad \Delta E_1 = \sum_{\beta} \frac{|\langle \psi_{2\beta} | V | \phi_A^0 \rangle|^2}{E - \epsilon_{2\beta}} - \left[\sum_{\beta} \frac{|\langle \psi_{2\beta} | V | \phi_A^0 \rangle|^2}{E - \epsilon_{2\beta}} \right]_{AV},$$

where $\epsilon_{2\beta}$ and $|\psi_{2\beta}\rangle$ are the eigenvalues and eigenvectors, respectively, corresponding to the 2p-2h excitations.

Since in the present scheme only the square of the corrections to the mean field, namely

$$(4.15) \quad (\delta E_1)^2 = [(\Delta E_1)^2]_{AV} - [(\Delta E_1)]_{AV}^2,$$

are meaningful, we are left with the task of evaluating

$$(4.16) \quad (\delta E_1)^2 = \left[\sum_{\beta\gamma} \frac{|\langle\psi_{2\beta}|V|\phi_A^0\rangle|^2 |\langle\psi_{2\gamma}|V|\phi_A^0\rangle|^2}{(E - \epsilon_{2\beta})(E - \epsilon_{2\gamma})} \right]_{AV} - \left[\sum_{\beta} \frac{|\langle\psi_{2\beta}|V|\phi_A^0\rangle|^2}{E - \epsilon_{2\beta}} \right]_{AV}^2.$$

Introducing an average excitation energy $\bar{\epsilon}_2$ for the 2p-2h states, the first term of the above equation can be recast as follows

$$(4.17) \quad \left[\sum_{\alpha,\beta} \frac{\langle\phi_A^0|V|\psi_{2\alpha}\rangle \langle\psi_{2\alpha}|V|\phi_A^0\rangle \langle\phi_A^0|V|\psi_{2\beta}\rangle \langle\psi_{2\beta}|V|\phi_A^0\rangle}{(E - \epsilon_{2\alpha})(E - \epsilon_{2\beta})} \right]_{AV} \\ \cong \left(\frac{1}{E - \bar{\epsilon}_2} \right)^2 \left[\sum_{\alpha,\beta} \langle\phi_A^0|V|\psi_{2\alpha}\rangle \langle\psi_{2\alpha}|V|\phi_A^0\rangle \langle\phi_A^0|V|\psi_{2\beta}\rangle \langle\psi_{2\beta}|V|\phi_A^0\rangle \right]_{AV}.$$

The random phase average of the quantity in the square brackets is next obtained using

$$(4.18) \quad \langle AA^* BB^* \rangle = \langle AA^* \rangle \langle BB^* \rangle + \langle A^* B \rangle \langle AB^* \rangle + \text{quartic terms};$$

quartic terms will be neglected.

Now, the first term $\langle AA^* \rangle \langle BB^* \rangle$ cancels exactly the second term in eq. (4.16). The evaluation of the second term in eq. (4.18) then yields

$$(4.19) \quad \begin{aligned} [(\delta E_1)^2] &= \left(\frac{1}{E - \bar{\epsilon}_2} \right)^2 \sum_{\alpha} [|\langle\phi_A^0|V|\psi_{2\alpha}\rangle|_{AV}^2]^2 \\ &\cong \frac{\overline{\Delta E_1}}{(E - \bar{\epsilon}_2)^2} \frac{1}{D_2} [|\langle\phi_A^0|V|\psi_2\rangle|_{AV}^2]^2 \end{aligned}$$

where $\overline{\Delta E_1}$, essentially given by ϵ , is the interval over which the energy average is carried out. The quantity D_2 is the energy distance between two neighboring 2p-2h states and, finally, ψ_2 is meant as a representative of the 2p-2h excitations.

We next consider ΔE_2 :

$$\begin{aligned} \Delta E_2 &= \langle\phi_A^0|V_{01}G_1V_{12}G_2V_{21}G_1V_{10}|\phi_A^0\rangle \\ &= \sum_{\alpha,\beta,\alpha'} \frac{\langle\phi_A^0|V|\psi_{2\alpha}\rangle}{E - \epsilon_{2\alpha}} \frac{\langle\psi_{2\alpha}|V|\psi_{4\beta}\rangle}{E - \epsilon_{4\beta}} \frac{\langle\psi_{4\beta}|V|\psi_{2\alpha'}\rangle}{E - \epsilon_{2\alpha'}} \langle\psi_{2\alpha'}|V|\phi_A^0\rangle \\ &\cong \frac{1}{(E - \bar{\epsilon}_2)^2} \frac{1}{(E - \bar{\epsilon}_4)} \left[\sum_{\alpha,\beta} \langle\phi_A^0|V|\psi_{2\alpha}\rangle \langle\psi_{2\alpha}|V|\phi_A^0\rangle \right] \end{aligned}$$

$$(4.20) \quad \cdot \langle \psi_{2\alpha} | V | \psi_{4\beta} \rangle \langle \psi_{4\beta} | V | \psi_{2\alpha} \rangle - \left[\sum_{\alpha, \beta} |\langle \phi_A^0 | V | \psi_{2\alpha} \rangle|^2 |\langle \psi_{2\alpha} | V | \psi_{4\beta} \rangle|^2 \right]_{AV}$$

where we have set $\alpha = \alpha'$, introduced an average energy $\bar{\epsilon}_4$ for the 4p–4h excitations and subtracted in the last line the average of ΔE_2 as written in the first line in order to achieve $\langle \Delta E_2 \rangle = 0$. Focussing now on the square of the modified $\Delta E_2 = \delta E_2$ we get

$$(4.21) \quad (\delta E_2)^2 \cong \frac{1}{(E - \bar{\epsilon}_2)^4} \frac{1}{(E - \bar{\epsilon}_4)^2} \left[\sum_{\alpha, \beta, \alpha', \beta'} \langle \phi_A^0 | V | \psi_{2\alpha} \rangle \langle \psi_{2\alpha} | V | \phi_A^0 \rangle \langle \psi_{2\alpha} | V | \psi_{4\beta} \rangle \langle \psi_{4\beta} | V | \psi_{2\alpha} \rangle \left\{ \langle \phi_A^0 | V | \psi_{2\alpha'} \rangle \langle \psi_{2\alpha'} | V | \phi_A^0 \rangle \langle \psi_{2\alpha'} | V | \psi_{4\beta'} \rangle \langle \psi_{4\beta'} | V | \psi_{2\alpha'} \rangle \right\} - \sum_{\alpha, \beta, \alpha', \beta'} [|\langle \phi_A^0 | V | \psi_{2\alpha} \rangle|^2 |\langle \phi_A^0 | V | \psi_{2\alpha'} \rangle|^2 |\langle \psi_{2\alpha} | V | \psi_{4\beta} \rangle|^2 |\langle \psi_{2\alpha'} | V | \psi_{4\beta'} \rangle|^2]_{AV} \right]_{AV}.$$

To calculate the average we again invoke eq. (4.18). As before the first term cancels the subtracted term and one is left with

$$(4.22) \quad (\delta E_2)^2 = \frac{1}{(E - \bar{\epsilon}_2)^4} \frac{1}{(E - \bar{\epsilon}_4)^2} \left[\sum_{\alpha, \beta} |\langle \phi_A^0 | V | \psi_{2\alpha} \rangle|^4 |\langle \psi_{2\alpha} | V | \psi_{4\beta} \rangle|^4 \right]_{AV}.$$

By comparing the above result with the one expressing $(\delta E_1)^2$, eq. (4.19), it follows

$$(4.23) \quad (\delta E_2)^2 = \frac{1}{(E - \bar{\epsilon}_2)^2} \frac{1}{(E - \bar{\epsilon}_4)^2} (\delta E_1)^2 |\langle \psi_2 | V | \psi_4 \rangle|^4$$

from where an expression for the *expansion parameter* is emerging, the expansion being however a finite one as previously emphasized. Likewise ψ_2 , in the above ψ_4 is describing a typical 4 particle–4 hole state.

The generalization of eq. (4.23) to an expression connecting $(\delta E_n)^2$ and $(\delta E_{n-1})^2$ is then easily found to read

$$(4.24) \quad [(\delta E_n)^2] = \frac{1}{(E - \bar{\epsilon}_{2n})^2} \frac{1}{(E - \bar{\epsilon}_{2n-2})^2} [(\delta E_{n-1})^2] [|\langle \psi_{2n-2} | V | \psi_{2n} \rangle|^4].$$

The results obtained up to this point have a general validity and can be applied to any system thus justifying the concept of mean field: indeed they have been deduced in the framework of the statistical theory. To further proceed a detailed description of the system one wishes to study is required: as an example we shall consider nuclear matter.

5. – Explicit expressions for the projection operator and the HF theory

We need now an explicit form for both the projection operator P and the NN interaction in order to obtain the true energy (E) and the *mean field* one (\bar{E}_0) (the shell model energy for a nucleus).

First we search for a suitable projection operator. Suppose, for this purpose, that for a given interaction the HF problem has been solved and let $\{\phi_{\text{HF}}^i\}$, with the index $i = 1, 2, \dots$, be the single particle wave functions of the HF orbitals, which form an orthonormal complete set for the Hilbert space of a one particle system. Out of the $\{\phi_{\text{HF}}^i\}$ we can build an infinite set of Slater determinants $\{\chi_{\text{HF}}^i\}$ which also form an orthonormal complete set for the Hilbert space of the nucleus.

A natural choice for the projection operator is then

$$(5.1) \quad P = \sum_{i=1}^M |\chi_{\text{HF}}^i\rangle\langle\chi_{\text{HF}}^i|$$

which, for $M = 1$, reduces to

$$(5.2) \quad P = |\chi_{\text{HF}}^1\rangle\langle\chi_{\text{HF}}^1|,$$

$|\chi_{\text{HF}}^1\rangle$ being the ground state HF determinant of the nuclear system. In the following for sake both of simplicity and of illustration we shall stick to the case $M = 1$, although generalizations to larger values of M , while cumbersome, should be worth exploring.

Notice that, with P given by (5.2), the wave functions $(P\Psi)$ and $\langle P\Psi$ turn out to be proportional to each other and to the ground state HF determinant $|\chi_{\text{HF}}^1\rangle$ as they must since the P space has only one member.

With P given by (5.1) and the mean field by $\bar{\mathcal{H}}$ (eq. (3.12)) one gets

$$(5.3) \quad \bar{E}_0 = E_{\text{HF}} + \langle\chi_{\text{HF}}^1|VQQV^\dagger|\chi_{\text{HF}}^1\rangle \frac{1}{\bar{E}_0 - E - \epsilon}$$

where

$$(5.4) \quad E_{\text{HF}} = \langle\chi_{\text{HF}}^1|H_{PP}|\chi_{\text{HF}}^1\rangle.$$

and

$$(5.5) \quad V = H \sqrt{\frac{\bar{E}_0 - \epsilon - E}{\bar{E}_0 - \epsilon - H_{QQ}}}$$

(see eq. (3.13)). Now writing $Q = 1 - P$ in Eq. (5.3), one obtains

$$(5.6) \quad \bar{E}_0 = E_{\text{HF}} + \frac{1}{\bar{E}_0 - E - \epsilon} \left\{ \langle\chi_{\text{HF}}^1|VV^\dagger|\chi_{\text{HF}}^1\rangle - |\langle\chi_{\text{HF}}^1|V|\chi_{\text{HF}}^1\rangle|^2 \right\}.$$

Let us now specify the above general expression to the case of a Fermi gas (infinite nuclear matter), for which $|\chi_{\text{HF}}^1\rangle = |F\rangle$, $|F\rangle$ being the wave function of a Fermi sphere of radius k_F (the Fermi wavenumber). In infinite nuclear matter the quantity in curly

brackets in eq. (5.6), which we refer to as the *variance* of the residual interaction V , simplifies considerably. Indeed is (the states $|n\rangle$ labelling the spectrum of the Fermi sphere)

$$\begin{aligned}
 \langle F|VV^\dagger|F\rangle &= \sum_n \langle F|V|n\rangle \langle n|V^\dagger|F\rangle \\
 &\cong \langle F|V|F\rangle \langle F|V^\dagger|F\rangle \\
 &\quad + \sum_{1p-1h} \langle F|V|1p-1h\rangle \langle 1p-1h|V^\dagger|F\rangle \\
 &\quad + \sum_{2p-2h} \langle F|V|2p-2h\rangle \langle 2p-2h|V^\dagger|F\rangle \\
 &= |\langle F|V|F\rangle|^2 + \beta^2,
 \end{aligned}
 \tag{5.7}$$

the piece related to the 1p-1h states essentially vanishing as required by the Brillouin theorem [7] and further terms in the right hand side being neglected because of the dominant two-body character of the residual interaction V . We indicate the variance of the residual interaction V , defined in eq. (5.6), with β^2 to remind that in nuclear matter such a quantity turns out to be positive. To perform a reliable evaluation of β^2 is of course quite hard because the residual effective interaction is an intricate operator: indeed, V is not only energy-dependent, but dependent upon the energy averaging procedure as well. In addition it is highly-non-linearly connected to the hamiltonian acting in the Q -space.

However, even in abeyance of an explicit expression for the variance, the insertion (5.7) into (5.6) leads to the important result

$$\bar{E}_0 \cong E_{\text{HF}} + \frac{\beta^2}{\bar{E}_0 - E - \epsilon},
 \tag{5.8}$$

which embodies all the energies characterizing our problem, namely the HF, the mean field (which would correspond to the shell model in a finite nucleus) and the exact one.

From (5.8) one sees that even if a residual interaction V is given and the associated HF energy E_{HF} is calculated, still the energy E of the system cannot be obtained since the mean field energy \bar{E}_0 remains to be fixed. However, if as a first orientation one sets $\bar{E}_0 \simeq E$, then the remarkable result

$$\bar{E}_0 \cong E_{\text{HF}} - \frac{\beta^2}{\epsilon}
 \tag{5.9}$$

follows, showing that the mean field energy is *lower* than the HF one, the parameter ϵ being of course positive. This result holds only if $|\bar{E}_0 - E| \ll \epsilon$.

6. – Occupancy of the mean field ground state

The projected wave function $P\Psi$ is just a component of the full wave function Ψ : how big a fraction of Ψ is contained in $P\Psi$ is obviously a quantity of much interest. To

estimate it let $\langle P\Psi|P\Psi\rangle = S^2$ and search for an equation for S^2 in terms of the other quantities characterizing our problem. Since

$$(6.1) \quad S^2 = 1 - \langle Q\Psi|Q\Psi\rangle$$

using eq. (3.18) for $(Q\Psi)$ one obtains

$$(6.2) \quad S^2 = 1 - \langle \Phi_0|V_{PQ} \left(\frac{1}{E - h_{QQ}} \right)^2 V_{QP}|\Phi_0\rangle.$$

Take now the Fermi gas model for Ψ , i. e. $|F\rangle$. Then $\Phi_0 = S|F\rangle$ and

$$(6.3) \quad S^2 = 1 - S^2 \langle F|V_{PQ} \left(\frac{1}{E - h_{QQ}} \right)^2 V_{QP}|F\rangle$$

which, with some algebra and neglecting the weak energy dependence of h_{QQ} , can be cast into the form

$$(6.4) \quad \begin{aligned} S^2 &\cong 1 + S^2 \langle F|V_{PQ} \frac{d}{dE} \left(\frac{1}{E - h_{QQ}} \right) V_{QP}|F\rangle \\ &= 1 + S^2 \left[\langle F|\frac{d}{dE} \left(V_{PQ} \frac{1}{E - h_{QQ}} V_{QP} \right) |F\rangle \right. \\ &\quad \left. - \langle F|\frac{dV_{PQ}}{dE} \frac{1}{E - h_{QQ}} V_{QP}|F\rangle - \langle F|V_{PQ} \frac{1}{E - h_{QQ}} \frac{dV_{QP}}{dE} |F\rangle \right]. \end{aligned}$$

Because

$$(6.5) \quad \frac{dV_{PQ}}{dE} = -V_{PQ} \frac{1}{2(\bar{E}_0 - \epsilon - E)}$$

and

$$(6.6) \quad \frac{dV_{QP}}{dE} = -V_{QP} \frac{1}{2(\bar{E}_0 - \epsilon - E)}$$

it follows that

$$(6.7) \quad \begin{aligned} S^2 &= 1 \\ &+ S^2 \left[\frac{d}{dE} \langle F|V_{PQ} \frac{1}{E - h_{QQ}} V_{QP}|F\rangle + \frac{1}{\bar{E}_0 - \epsilon - E} \langle F|V_{PQ} \frac{1}{E - h_{QQ}} V_{QP}|F\rangle \right] \end{aligned}$$

or, by virtue of (3.21), that

$$(6.8) \quad \begin{aligned} S^2 &= 1 + S^2 \left[\frac{d}{dE} (E - \bar{E}_0) + \frac{E - \bar{E}_0}{\bar{E}_0 - \epsilon - E} \right] \\ &= 1 - S^2 \left(\frac{d\bar{E}_0}{dE} + \frac{\epsilon}{\bar{E}_0 - \epsilon - E} \right). \end{aligned}$$

The explicit dependence of \bar{E}_0 upon E is provided by (5.8) which, when inverted, yields

$$(6.9) \quad \bar{E}_0 = \frac{1}{2} \left[E + \epsilon + E_{\text{HF}} \pm \sqrt{(E + \epsilon - E_{\text{HF}})^2 + 4\beta^2} \right],$$

the minus sign in front of the square root being taken, since for vanishing residual interaction ($\beta^2 = 0$) the mean field should reduce to the HF field ($\bar{E}_0 = E_{\text{HF}}$). Hence

$$(6.10) \quad \frac{d\bar{E}_0}{dE} = \frac{1}{2} + \frac{1}{2} \frac{E_{\text{HF}} - E - \epsilon - 2d\beta^2/dE}{\sqrt{(E_{\text{HF}} - E - \epsilon)^2 + 4\beta^2}},$$

which, inserted into (6.8), finally leads to

$$(6.11) \quad S^2 = \left[\frac{3}{2} + \frac{1}{2} \frac{E_{\text{HF}} - E - \epsilon - 2d\beta^2/dE}{\sqrt{(E_{\text{HF}} - E - \epsilon)^2 + 4\beta^2}} + \frac{\epsilon}{\bar{E}_0 - E - \epsilon} \right]^{-1} \\ = \left[\frac{3}{2} + \frac{1}{2} \frac{E_{\text{HF}} - E - \epsilon - 2d\beta^2/dE}{\sqrt{(E_{\text{HF}} - E - \epsilon)^2 + 4\beta^2}} + \epsilon \frac{\bar{E}_0 - E_{\text{HF}}}{\beta^2} \right]^{-1},$$

namely to the equation we were looking for. We thus see that the occupancy of the mean field ground state wave function S^2 is expressed in terms of the HF, mean field and exact energies, with an additional dependence upon the energy averaging parameter ϵ , the residual interaction β^2 and the derivative of the latter with respect to the energy. Since all these quantities are either explicitly evaluated (E_{HF} , \bar{E}_0 and E) or fixed, in principle, both by the experiment (ϵ) and by our theoretical framework (β^2), eq. (6.11) provides an important check for the consistency of our approach, also because independent estimates on the depletion of the ground state wave function are available for lead, a nucleus for which nuclear matter is a reliable model, the present approach can be tested to some extent against the experiment.

7. – A simple model for nuclear matter

To gain an insight into the effectiveness and self-consistency of the formalism described above we develop here a simple, schematic model. The model relies on two equations which, for sake of convenience, are here repeated. The first of these is eq. (5.8)

$$\bar{E}_0 \cong E_{\text{HF}} + \frac{\beta^2}{\bar{E}_0 - E - \epsilon}.$$

The second is obtained from Eq. (4.19), i. e.

$$[(\delta E_1)^2] = \left(\frac{1}{E - \bar{\epsilon}_2} \right)^2 \sum_{\gamma} [|\langle \phi_A^0 | V | \psi_{2\gamma} \rangle|_{AV}^2]^2,$$

which we heuristically approximate (admittedly roughly) by

$$(7.1) \quad [(\delta E_1)^2] = \frac{\beta^4}{(E - \alpha\epsilon)^2},$$

where α is a parameter which estimates the average energy of the 2p–2h states involved (we set $\bar{\epsilon}_2 = \alpha\epsilon$ for convenience) and *importantly* helps to correct for the poor approximation of β^4 for the sum over the 2p–2h states γ in eq. (4.19). Taking the square root,

from eq. (7.1) it follows

$$(7.2) \quad E - \bar{E}_0 = \pm \frac{\beta^2}{E - \alpha\epsilon}.$$

If now one combines eq. (5.8) and (7.2), using the upper sign (plus) one obtains a lower bound E_l for the energy of nuclear matter, while using the lower sign (minus) one obtains an upper bound E_u . Clearly, one gets also differing values of the mean field energy to be denoted by \bar{E}_0^l and \bar{E}_0^u , respectively. We ask then whether there is a choice of ϵ , α and β such that

- i) the two mean field energies \bar{E}_0^l and \bar{E}_0^u turn out to be the same over a range of densities (or of Fermi momenta k_F) of significance for nuclear matter;
- ii) the “experimental” values of the binding energy, saturation density and compression modulus of nuclear matter, namely $B.E./A = -16$ MeV, $k_F = 1.36$ fm $^{-1}$ and $K = 23$ or 16 MeV (the two values refer to a hard and a soft equation of state, respectively) [8] are accounted for in a sense to be later specified;
- iii) the spectroscopic factor given by eq. (6.11) turns out to be less than one both on the lower bound (where its value S_l is associated with E_l) and on the upper one (where its value S_u is associated with E_u) and furthermore not too much at variance with existing estimates.

To accomplish this program we proceed by first eliminating \bar{E}_0^u and \bar{E}_0^l from eq. (5.8) and (7.2). This yields an equation for β_l^2 and β_u^2 :

$$(7.3) \quad \beta_l^2 = \frac{E_l - \alpha\epsilon}{2} \left\{ [2E_l - \epsilon(\alpha + 1) - E_{\text{HF}}] - \sqrt{[2E_l - \epsilon(\alpha + 1) - E_{\text{HF}}]^2 + 4\epsilon(E_l - E_{\text{HF}})} \right\},$$

for the lower bound, and

$$(7.4) \quad \beta_u^2 = \frac{E_u - \alpha\epsilon}{2} \left\{ [E_{\text{HF}} - \epsilon(\alpha - 1)] - \sqrt{[E_{\text{HF}} - \epsilon(\alpha - 1)]^2 + 4\epsilon(E_u - E_{\text{HF}})} \right\},$$

for the upper bound. Note that the solutions with the plus sign in front of the square root have been discarded, since they lead to an incorrect limit as $\epsilon \rightarrow 0$. The energy derivatives of (7.3) and (7.4) are then easily obtained, namely

$$(7.5) \quad \frac{d\beta_l^2}{dE} = \frac{1}{2} \left\{ [2E_l - \epsilon(\alpha + 1) - E_{\text{HF}}] - \sqrt{[2E_l - \epsilon(\alpha + 1) - E_{\text{HF}}]^2 + 4\epsilon(E_l - E_{\text{HF}})} \right\} \\ + (E_l - \alpha\epsilon) \left\{ 1 - \frac{2E_l - \alpha\epsilon - E_{\text{HF}}}{\sqrt{[2E_l - \epsilon(\alpha + 1) - E_{\text{HF}}]^2 + 4\epsilon(E_l - E_{\text{HF}})}} \right\},$$

for the lower bound, and

$$(7.6) \quad \frac{d\beta_u^2}{dE} = \frac{1}{2} \left\{ [E_{\text{HF}} - \epsilon(\alpha - 1)] - \sqrt{[E_{\text{HF}} - \epsilon(\alpha - 1)]^2 + 4\epsilon(E_u - E_{\text{HF}})} \right\} \\ - (E - \alpha\epsilon) \frac{\epsilon}{\sqrt{[E_{\text{HF}} - \epsilon(\alpha - 1)]^2 + 4\epsilon(E_u - E_{\text{HF}})}},$$

for the upper bound, These, when inserted into (6.11), provide an expression for the spectroscopic factor, associated with the lower and the upper bound, respectively, in terms of E_l (E_u), \bar{E}_0^l (\bar{E}_0^u), E_{HF} , α and ϵ .

From eqs. (7.3) and (7.4) we note that should appropriate values for E_l (or E_u) and E_{HF} be available one would then obtain a value of β_l^2 and β_u^2 for given values of α and ϵ . These values of β^2 can in turn be used in eq. (5.8) to obtain \bar{E}_0^l and \bar{E}_0^u . We agree that a solution of eq. (5.8) and (7.2) occurs when these upper and lower bounds for the mean field are equal.

The problem remains of how to choose meaningful values for E_l and E_u and how to fix E_{HF} . The first of these questions is answered by taking

$$(7.7a) \quad E_l = \mathcal{E} - W/2$$

$$(7.7b) \quad E_u = \mathcal{E} + W/2,$$

where the energy

$$(7.8) \quad \mathcal{E} = [-16 + 39.5(k_F - 1.36)^2]\text{MeV}$$

incorporates the present knowledge on the ground state energy of nuclear matter, quoted in ii), as extrapolated from finite nuclei and W should be viewed as the fluctuation energy.

The value of E_{HF} as a function of k_F is obtained from an assumed, schematic two-body interaction (Appendix A). The resulting Hartree-Fock energy for a Fermi gas is shown in Fig. 2 and, for a reduced scale in k_F , in Fig. 4. From Fig. 2 we see that for our simple two-body force a minimum of the binding energy as a function of k_F occurs at $k_F = 1.78 \text{ fm}^{-1}$, the corresponding energy per particle being -7.3 MeV .

The potential can also be used to calculate the bare value of β^2 (Appendix B). Obtained values are shown in Tables I, II and III: they range from 4.5×10^4 to $7.0 \times 10^4 \text{ MeV}^2/\text{nucleon}$.

In discussing our findings it help first to observe that actually, for a given fluctuation energy W , a whole set of values for the parameters α and ϵ exist such to satisfy the requirement i), namely $\bar{E}_0^l = \bar{E}_0^u$, at $k_F = 1.36 \text{ fm}^{-1}$. They lie on the curves displayed in Fig. 3. Furthermore α and ϵ should also be such to fulfill the constraint

$$(7.9) \quad \bar{\epsilon}_2 = \alpha\epsilon \geq 20\text{MeV},$$

which represents a fair estimate of the lower limit for the excitation energy of the 2p-2h states. As a consequence the acceptable values for α and ϵ are then restricted to the domain to the right of the dotted line in Fig. 3.

In this region α and ϵ should be selected in such a way to comply with the requirements ii) and iii). This turns out to be possible and our results are shown in Fig. 4 and Tables I, II and III.

However there one sees that as one moves away from $k_F = 1.36 \text{ fm}^{-1}$ the equality between \bar{E}_0^l and \bar{E}_0^u is no longer exactly satisfied, although the two mean fields remain rather close to each other at least in the range $0.9 \leq k_F \leq 1.5 \text{ fm}^{-1}$ providing the values of W are “moderate”. Indeed, quantitatively, requirement i) can be reasonably satisfied

in the above quoted range of k_F for $W \leq 4$ MeV only. For larger fluctuation energies \bar{E}_0^l and \bar{E}_0^u differ too much: an orientation on the size of the error around the mean field is thus obtained.

Concerning the energy averaging parameter, ϵ , it turns out to be about 1 MeV larger than W for all the cases listed in the tables. The values of the spectroscopic factors S_l and S_u are quite stable, while the values of the effective interaction β_l^2 and β_u^2 grow rapidly with increasing W . Most importantly, their values differ by *three orders of magnitude* from the bare β^2 . This occurrence partly stems from the renormalization (see eqs. (3.13) and (3.14)) induced by the energy averaging. The random phase averaging also gives rise to a reduction in the magnitude of the residual interaction. Calculations, while rough, still indicate that these two effects appear indeed sufficient to produce the observed sharp reduction.

8. – Closing comments

The statistical theory of the mean field presented in sections 1–4 is based on two propositions. It is assumed that the mean field is the slowly varying component of the nuclear interaction, which can be obtained by taking an appropriate energy average. Secondly it is suggested that the matrix elements of the residual interaction are random, so that their average is zero. A formalism incorporating these ideas, borrowed from statistical reaction theory, was developed and explicit expressions for the mean field and the “fluctuation” away from the mean field was obtained.

Next an expansion of the fluctuation energy in terms of increasing excitation complexity leads, after averaging, to formulas for the corresponding contributions to fluctuation energy. Worth emphasizing is the novelty of this approach: indeed the corrections to the mean field ground state energy of the system rather than being ordered according to the power of a coupling constant or according to the number of hole lines occurring in a given diagram, as it is done in standard theories, are here organized in terms of the complexity of the Q -space states reached via the residual interaction acting among the constituents of the system.

The presentation of this approach was followed in sect. 5, 6 and 7 by a simplified version and by a schematic model. The most remarkable result emerging from the analysis of the model has been the sharp reduction of the effective strength of the residual interaction. But in addition is the overall reasonableness of our results which is quite encouraging.

Of course much remains to be done. The quantitative connection with the underlying nuclear forces has not been exhibited. The evaluation of the matrix elements for finite nuclei was not carried out and a better understanding of the energy average needs to be achieved. We have so far only a schematic model. What is needed is a complete and thorough evaluation which, on the basis of the obtained results can be expected with confidence to be successful.

Application to excited states is also indicated. In this case the smoothing function used in reaction theory can be used instead of the one of section 4. This would lead to

a complex mean field and the excited states would have a width corresponding to the probability of the splitting of the state by the residual interaction. The width would then measure the extent of the splitting.

Of crucial importance is also to test the rate of convergence of the complexity expansion. Three elements are expected to favour in general a fast convergence rate for the latter:

- i) the energy averaging parameter ϵ , if it is large enough;
- ii) the subtraction of the average interaction from the bare one;
- iii) the reduced overlap between the ground state and the complex wave functions belonging to the Q -space.

A detailed investigation will be invaluable in shedding light on this issue. In this connection calculations are presently in progress.

APPENDIX A.

The bare interaction and the HF theory

To implement the program outlined in Section 7 we need a NN interaction to fix the HF energy of nuclear matter. For illustrative purposes we choose the following simple NN interaction

$$(A.1) \quad \mathcal{V}(r) = g_A \frac{e^{-\mu_A r}}{r} - g_B \frac{e^{-\mu_B r}}{r} \frac{1 + P_x}{2},$$

which embodies a short-range repulsion in the first term and an intermediate range attraction in the second one (all the parameters are positive and we require $\mu_A > \mu_B$). The latter is taken of Majorana type, hence the occurrence of the exchange operator

$$(A.2) \quad P_x = \frac{1 + \vec{\tau}_1 \cdot \vec{\tau}_2}{2} \frac{1 + \vec{\sigma}_1 \cdot \vec{\sigma}_2}{2}$$

built out of the spin $\vec{\sigma}$ and isospin $\vec{\tau}$ operators. Thus (A.1) contains the main features needed to account for the saturation of the nuclear forces, reflected in the existence of a minimum in the binding energy per particle ($B.E./A$) versus k_F curve, but for the tensor force that here, for the sake of simplicity, is neglected.

The HF energy for the interaction (A.1) is easily worked out and leads to the following expression for the binding energy per particle [9]

$$(A.3) \quad \begin{aligned} \frac{B.E.}{A} = & \frac{3}{5} \frac{\hbar^2 k_F^2}{2m} + \frac{\rho}{2} \left(4\pi \frac{g_A}{\mu_A^2} - \frac{3\pi}{2} \frac{g_B}{\mu_B^2} \right) - \frac{3k_F}{4\pi} \left(g_A + \frac{3}{2} g_B \right) \\ & + \frac{1}{8\pi k_F} \left(g_A \mu_A^2 + \frac{3}{2} g_B \mu_B^2 \right) \\ & + \frac{1}{\pi} \left[g_A \mu_A \arctan \left(\frac{2k_F}{\mu_A} \right) + \frac{3}{2} g_B \mu_B \arctan \left(\frac{2k_F}{\mu_B} \right) \right] \end{aligned}$$

$$-\frac{1}{8\pi k_F} \left[g_A \mu_A^2 \left(3 + \frac{\mu_A^2}{4k_F^2} \right) \log \left(1 + \frac{4k_F^2}{\mu_A^2} \right) + \frac{3}{2} g_B \mu_B^2 \left(3 + \frac{\mu_B^2}{4k_F^2} \right) \log \left(1 + \frac{4k_F^2}{\mu_B^2} \right) \right],$$

where $\varrho = 2k_F^3/3\pi^2$.

The parameters characterizing the potential (A.1) might be fixed, for example, by accounting for the “experimental” nuclear matter values previously quoted. One succeeds in doing so with the following choice

$$\begin{aligned} \text{(A.4a)} \quad \mu_A &= 3.43 \text{ fm}^{-1}, & \mu_B &= 1.63 \text{ fm}^{-1} \\ \text{(A.4b)} \quad g_A &= 2460 \text{ MeV fm}, & g_B &= 898 \text{ MeV fm}. \end{aligned}$$

Worth noticing is that the range of the repulsion obtained with the fitting procedure is rather close to the one associated with the exchange of a ω meson (3.97 fm^{-1}), whereas the range of the attraction turns out to be intermediate to the one arising from the exchange of a pion and of a σ meson (0.71 fm^{-1} and 2.79 fm^{-1} , respectively).

Here, we rather prefer to choose the parameters in such a way to have too little binding energy at too large a density in the HF frame, in order to conform to a shortcoming common to many nuclear matter calculations, the purpose being to ascertain whether the present theory is capable to improve upon the HF results. Of course, there exists a variety of ways for reaching this scope: In view of the rather realistic values of the ranges μ_A and μ_B (see (A.4a)) we change the coupling constants. We thus take, as a rather extreme example,

$$\text{(A.5)} \quad g_A = 740 \text{ MeV fm}, \quad g_B = 337 \text{ MeV fm},$$

which, together with the values for μ_A and μ_B given in (A.4a), yields a minimum of -7.30 MeV for the binding energy at $k_F = 1.78 \text{ fm}^{-1}$, as it can be seen in Fig. 2, where the HF energy is displayed as a function of the density.

APPENDIX B.

Vacuum \rightarrow 2p–2h matrix element

In the language of second quantization the matrix element we have to calculate reads

$$\text{(B.6)} \quad \beta^2 = \sum_{\substack{\text{spin} \\ \text{isospin}}} \sum_{\substack{k_1, k_2 < k_F \\ |\vec{k}_1 + \vec{q}|, |\vec{k}_2 - \vec{q}| > k_F}} \langle \vec{k}_1, \vec{k}_2 | V | \vec{k}_1 + \vec{q}, \vec{k}_2 - \vec{q} \rangle \langle \vec{k}_1 + \vec{q}, \vec{k}_2 - \vec{q} | V | \vec{k}_1, \vec{k}_2 \rangle,$$

which, with standard manipulations, can be transformed into

$$\begin{aligned} \beta^2 &= A \frac{2}{\pi \varrho} \int \frac{d\vec{k}_1}{(2\pi)^3} \frac{d\vec{k}_2}{(2\pi)^3} d\vec{q} \Theta \left(|\vec{k}_1 + \vec{q}| - k_F \right) \Theta \left(|\vec{k}_2 - \vec{q}| - k_F \right) \Theta \left(k_F - k_1 \right) \\ &\quad \times \Theta \left(k_F - k_2 \right) \left\{ 32g_A^2 \frac{1}{(\mu_A^2 + q^2)^2} + 12g_B^2 \frac{1}{(\mu_B^2 + q^2)^2} - 24g_A g_B \frac{1}{\mu_A^2 + q^2} \frac{1}{\mu_B^2 + q^2} \right\} \end{aligned}$$

$$(B.7) \quad -8g_A^2 \frac{1}{\mu_A^2 + q^2} \frac{1}{\mu_A^2 + |\vec{k}_1 - \vec{k}_2 + \vec{q}|^2} + 12g_B^2 \frac{1}{\mu_B^2 + q^2} \frac{1}{\mu_B^2 + |\vec{k}_1 - \vec{k}_2 + \vec{q}|^2} \\ -12g_A g_B \left(\frac{1}{\mu_A^2 + q^2} \frac{1}{\mu_B^2 + |\vec{k}_1 - \vec{k}_2 + \vec{q}|^2} + \frac{1}{\mu_B^2 + q^2} \frac{1}{\mu_A^2 + |\vec{k}_1 - \vec{k}_2 + \vec{q}|^2} \right) \Bigg\}$$

(the matrix elements in eq. (B.6) are understood to be antisymmetrized). The above 9-dimensional integral, by appropriate transformations, can be reduced to a combination of 2- and 4-dimensional integrals, which can be numerically evaluated, yielding the results quoted in Tables I–III. These values turn out to be helpful in performing a comparison with later findings.

The direct contributions, namely the first three terms in the right hand side of (B.7), are analytically evaluated. Considering, e. g., the third one, one gets

$$(B.8) \quad A \frac{2}{\pi \varrho} \int \frac{d\vec{k}_1}{(2\pi)^3} \frac{d\vec{k}_2}{(2\pi)^3} d\vec{q} \Theta(|\vec{k}_1 + \vec{q}| - k_F) \Theta(|\vec{k}_2 - \vec{q}| - k_F) \Theta(k_F - k_1) \Theta(k_F - k_2) \\ \times \left\{ \frac{1}{\mu_A^2 + q^2} \frac{1}{\mu_B^2 + q^2} \right\} \\ = A \frac{\varrho}{4k_F} \left\{ \frac{1}{\tilde{\mu}_A^2 - \tilde{\mu}_B^2} (\tilde{\mu}_A \arctan \tilde{\mu}_A - \tilde{\mu}_B \arctan \tilde{\mu}_B) \right. \\ + \frac{9}{5} + \frac{17}{12} (\tilde{\mu}_A^2 + \tilde{\mu}_B^2) + \frac{1}{4} (\tilde{\mu}_A^4 + \tilde{\mu}_B^4) + \frac{1}{4} \tilde{\mu}_A^2 \tilde{\mu}_B^2 \\ \left. + \frac{1}{4} \frac{1}{\tilde{\mu}_A^2 - \tilde{\mu}_B^2} \left[\tilde{\mu}_B^3 (3 + \tilde{\mu}_B^2)^2 \arctan \left(\frac{1}{\tilde{\mu}_B} \right) - \tilde{\mu}_A^3 (3 + \tilde{\mu}_A^2)^2 \arctan \left(\frac{1}{\tilde{\mu}_A} \right) \right] \right\},$$

where $\tilde{\mu}_{A,B} = \mu_{A,B}/2k_F$.

The other terms are obtained with obvious substitutions.

The exchange contribution is somewhat more involved. As an example we consider (again the other pieces are obtained through obvious interchanges)

$$(B.9) \quad \beta_{\text{exc}}^2 = A \frac{2}{\pi \varrho} \int \frac{d\vec{k}_1}{(2\pi)^3} \frac{d\vec{k}_2}{(2\pi)^3} d\vec{q} \\ \Theta(|\vec{k}_1 + \vec{q}| - k_F) \Theta(|\vec{k}_2 - \vec{q}| - k_F) \Theta(k_F - k_1) \Theta(k_F - k_2) \\ \times \left\{ \frac{1}{\mu_A^2 + q^2} \frac{1}{\mu_B^2 + |\vec{k}_1 - \vec{k}_2 + \vec{q}|^2} \right\} \\ = \frac{A}{(2\pi^2)} \frac{2}{\pi^2 \varrho} \sum_{l=0}^{\infty} (-1)^l (2l+1) \int_0^{\infty} dq \frac{q^2}{\mu_A^2 + q^2} \int_0^{\infty} dr r e^{-\mu_B r} j_0(qr) \\ \times \left[\int_0^{k_F} dk k^2 j_l(kr) \int_{-1}^1 dx P_l(x) \Theta(q^2 + k^2 + 2qkx - k_F) \right]^2,$$

where a partial wave expansion, in terms of the Legendre polynomials P_l and the spherical Bessel functions j_l , has been performed. The above integral, for $q \geq 2k_F$ is easily

calculated yielding

$$(B.10) \quad \beta_{\text{exc}}^2 = A \frac{2}{\pi^4 \varrho} \int_0^\infty dr \frac{e^{-\mu_B r}}{r^6} \left(\sin(k_F r) - k_F r \cos(k_F r) \right)^2 \\ \times \int_{2k_F}^\infty dq \frac{q}{\mu_A^2 + q^2} \sin(qr).$$

The remaining integrals are numerically evaluated.

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TABLE I. – In the second row the value of \bar{E}_0 as a function of k_F is reported; in the third row one finds the bare value of β^2 , as given by eq. (B.7), while in the fourth and fifth rows the estimated renormalized values on the lower and upper bounds, respectively; in the last two rows, the corresponding values of the spectroscopic factor are displayed. The values in this table correspond to $W = 2$ MeV, $\epsilon = 3$ MeV and $\bar{\epsilon}_2 = 22$ MeV.

k_F (fm $^{-1}$)	1.2	1.36	1.5
\bar{E}_0 (MeV)	-15.3÷-15.5	-16.3	-15.6÷-15.2
β^2	$4.5 \cdot 10^4$	$5.2 \cdot 10^4$	$7.0 \cdot 10^4$
β_l^2 (MeV 2 /nucleon)	27.0	26.3	22.8
β_u^2	54.2	49.0	36.4
S_l	0.32	0.32	0.30
S_u	0.64	0.62	0.58

TABLE II. – As in table I: The values in this table correspond to $W = 3$ MeV, $\epsilon = 4$ MeV and $\bar{\epsilon}_2 = 20$ MeV.

k_F (fm $^{-1}$)	1.2	1.36	1.5
\bar{E}_0 (MeV)	-15.5÷-15.9	-16.6	-15.9÷-15.2
β^2	$4.5 \cdot 10^4$	$5.2 \cdot 10^4$	$7.0 \cdot 10^4$
β_l^2 (MeV 2 /nucleon)	36.1	35.3	30.7
β_u^2	79.6	69.9	49.8
S_l	0.34	0.33	0.32
S_u	0.69	0.67	0.63

TABLE III. – As in table I: The values in this table correspond to $W = 4$ MeV, $\epsilon = 5$ MeV and $\bar{\epsilon}_2 = 19$ MeV.

k_F (fm $^{-1}$)	1.2	1.36	1.5
\bar{E}_0 (MeV)	-15.7÷-16.4	-16.8	-16.1÷-15.2
β^2	$4.5 \cdot 10^4$	$5.2 \cdot 10^4$	$7.0 \cdot 10^4$
β_l^2 (MeV 2 /nucleon)	45.7	44.7	39.2
β_u^2	108.6	91.9	62.7
S_l	0.35	0.34	0.33
S_u	0.73	0.72	0.66

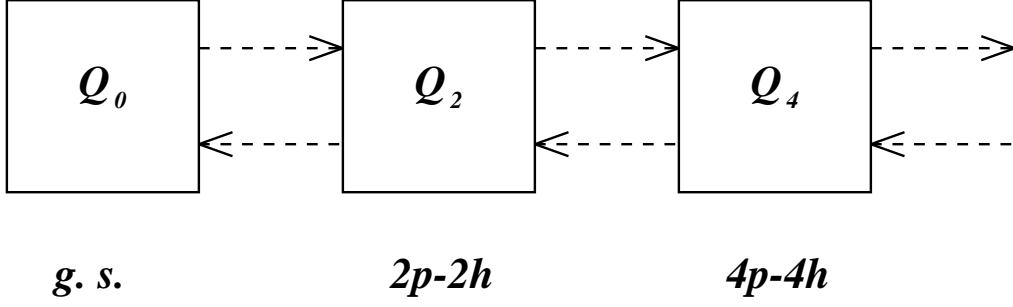


Fig. 1. – The partition of the Hilbert space of nuclear matter in sets of increasing complexity. The first box on the left defines the P -space, the second one embodies the simplest states in the Q -space and so on.

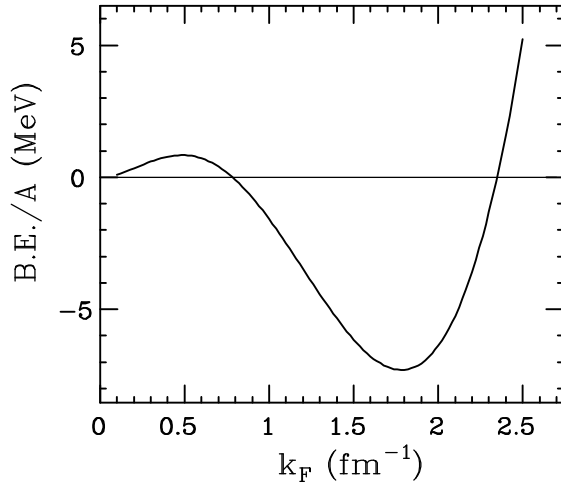


Fig. 2. – The binding energy per particle in the HF approximation (formula (A.3) of the text) for the potential (A.1) and the parameters given by (A.5).

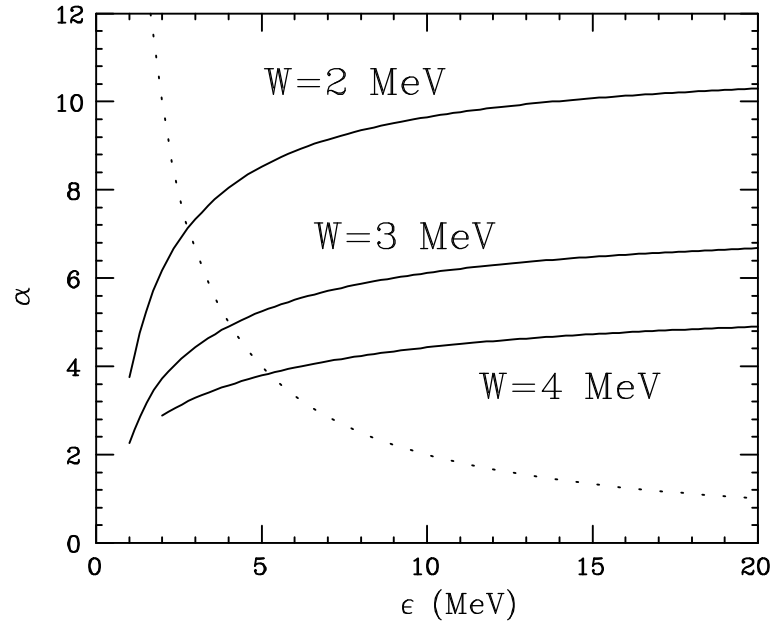


Fig. 3. – The loci corresponding to $\bar{E}_0^l = \bar{E}_0^a$ in the plane (α, ϵ) , for $W = 2, 3$ and 4 MeV. Also shown (dotted line) is the curve along which the average energy of the 2p–2h excitations is 20 MeV

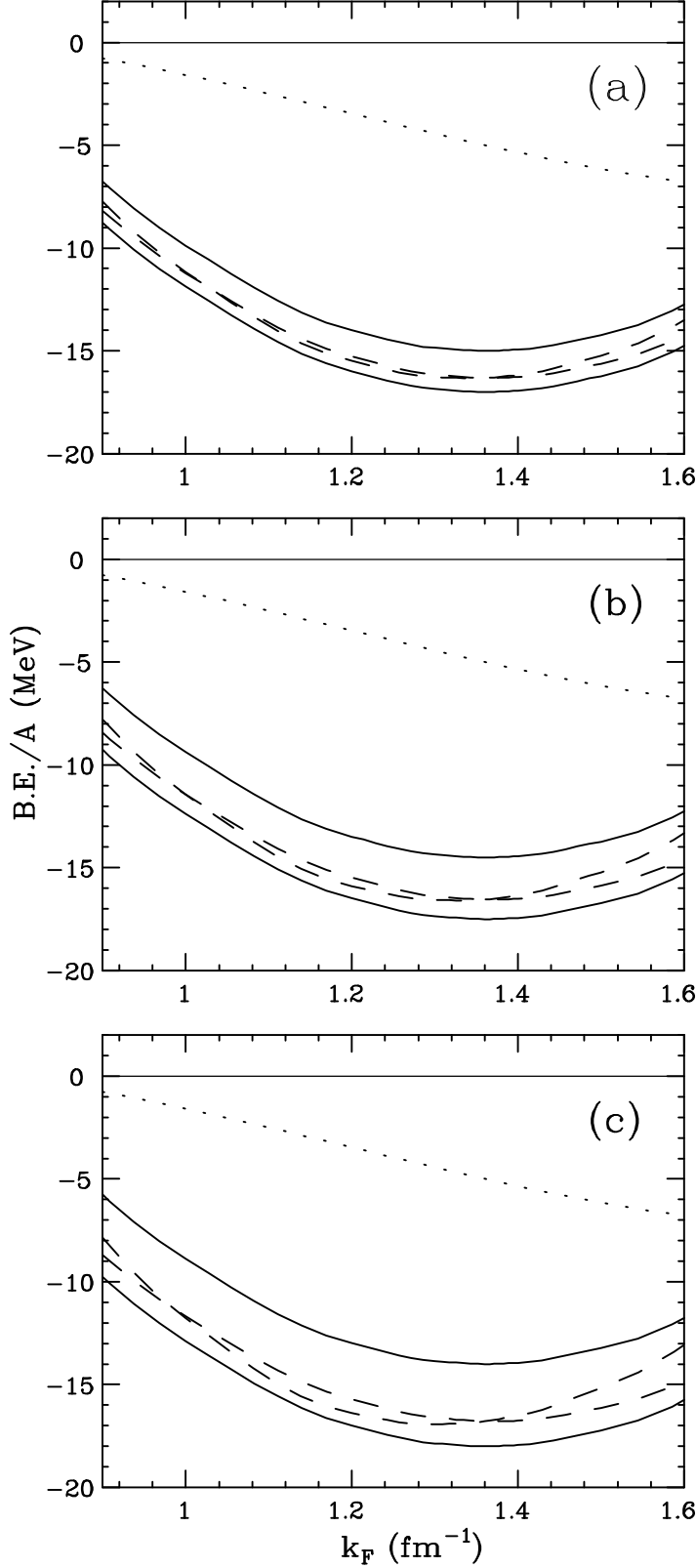


Fig. 4. – The binding energy per particle in the HF approximation (dot) and in the present approach: The solid lines represent the lower and upper bounds of fluctuations for the energy E , whereas the dashed lines give the corresponding mean field energies. (a): $W = 2$ MeV, $\epsilon = 3$ M.V., $\gamma = 7.2$, $\beta = 0.2$ M.V.; (b): $W = 7.2$ M.V., $\epsilon = 3$ M.V., $\gamma = 4.2$, $\beta = 0.2$ M.V.; (c): $W = 4$